Simultaneous-Equation Regression Analysis of Sales and Advertising*

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Summary—A simultaneous-equation regression model is developed and tested against aggregative time series data. The test involves the model as well as hypotheses regarding the structural parameters. Restrictions on the structural parameters imply limits on the parameters of the reduced-form equations. Estimates of the reduced-form coefficients must fall within the implied intervals in order for the model to be in agreement with the data.

Analysis of the influence of advertising on sales is a difficult and dangerous undertaking. The dangers are largely those associated with the fact that it is possible to obtain a good fit to the data with a great variety of statistical models and assumptions. Most previous studies of advertising and sales have focused primarily on estimation rather than testing, or they have relied upon goodness of fit as a criterion for choosing among alternative models. Broadly speaking, the difficulties involved in measuring the influences of advertising may be separated into three major categories: (1) the problem of isolating the effects of advertising from the many other variables which affect sales, (2) the problem of measuring the quantity of advertising, taking into account the fact that advertising dollar expenditures reflect alternative choices of media, psychological appeals, and copy, and, (3) the problem of identifying the relationship which reflects the influence of advertising upon sales—the so-called identification problem. In spite of the difficulties and dangers, the profit potential relative to the costs for those industries in which advertising is a major influence on sales easily justifies the testing of hypotheses about advertising and sales.

There have been a number of interesting single-equation regression studies of time series sales and advertising data including those by Telsner (1962a, b), Palda (1964), Weinberg (1961) and Vidale and Wolfe (1957). Explicit tests of hypotheses derived from models are noticeably rare in the literature, however. The same can be said of attempts to deal with the identification problem. A previous study by Bass (1967) was the first known application of the scientific concept of predictive testing and simultaneous-equation relations to sales and advertising data.

In this paper we shall develop and test a model against aggregative sales and advertising data, the form in which data are more commonly available to management. The model must

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*This research was supported by a grant from Leo Burnett Company—Advertising, Inc. We are especially grateful to Seymour Banks and John Couison for their support and advice. The responsibility for this work and for the conclusions is ours alone.
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pass a rather severe test in order to be found to be in agreement with the data. Furthermore, the model is formulated to take into account the simultaneous nature of the relationship between sales and advertising, a serious omission in previous studies. There is general agreement that not only is sales influenced by advertising, but advertising is also influenced by sales. Advertising decision rules, whether rigid or flexible, certainly take sales into account. Single-equation regression models are therefore seriously deficient with respect to the identification of the advertising sales, sales-advertising relationships. We shall formulate a multiple-equation regression model and test explicit propositions about simultaneous relationships between sales and advertising.

CONCEPT OF THE PREDICTIVE TEST

Since tests of theories utilizing well established principles of scientific methodology are notably sparse in statistical studies in economics and marketing, it may not be inappropriate to discuss them briefly here. It is important to note at the outset that no empirical test can prove that a theory is true. A theory may be shown to be in good agreement with the empirical evidence, but since it is not possible to test all of the infinitely many alternative theories, some of which may agree equally well with the data, the failure of a particular theory to be rejected is far from conclusive proof that it is true. It is for this reason that it is especially important to develop tests which are not easily satisfied. Unsuccessful predictions can, in principle, be conclusive with respect to the truth of a theory. It is important, however, to examine carefully the underlying statistical assumptions and background conditions regardless of the outcome of the test.

The concept of the predictive test within the framework of simultaneous-equation regression has been cogently established by Basmann (1963, 1965a). The theoretical foundations and empirical studies by Basmann (1964, 1965a) provide the basis for the methodological structure for this study. As indicated by Basmann '...the testing of theoretical premises about an economic parameter is logically prior to its estimation'.

In addition to a system of structural relations, an explanatory marketing model is comprised of theoretical marketing premises and justifiable factual statements of initial conditions. A set of prediction statements that attribute definite probabilities to specified observable marketing events is deduced from the model. Since the model can be proven false by unsuccessful predictions, close attention must be given to substantiating statements which claim that external influences are negligible during the historical period analyzed.

A statement of initial conditions is a combination of three statements. One statement specifies the observed values of the exogenous variable explicitly included in the structural relations of the model. Another statement specifies the statistical distribution of the random disturbances explicitly included in the structural relations. In addition, one statement asserts that relevant external conditions stay approximately constant during the historical period under consideration.

The predictive test of an explanatory marketing model is implemented by specification of an observable event (the critical region for the test) which has a probability of occurring that is very small if the conjunction of initial conditions and economic premises is true. Occurrence of this theoretically improbable event implies that at least one, and maybe only one, of the assumptions about the initial conditions and marketing premises is discredited. If factual
investigation justifies the statement of initial conditions, then at least one, and maybe only one, of the premises is discredited.

It is important to note that a forecast and a prediction are not synonymous. Important and pragmatic statements about future occurrences can often be made without deducing these statements from initial conditions with the aid of a model. Naive, non-explanatory models frequently yield good forecasts. A forecast is an extrapolation from statistical parameter estimates obtained in one historical period to observations generated in another historical period. While, as a matter of interest, we shall develop a forecast from the model to be tested, it is not the basis for accepting or rejecting the model.

THE DATA AND INDUSTRY BACKGROUND

Anticipating the statistical results, we can report that the predictions have survived the empirical test. While successful predictions do not provide conclusive proof of the truth of the model from which they were derived, it may be useful to make them public and to describe, in general terms, the background and setting of the industry. This discussion about the origins of premises is not meant to be an argument defending them as self-evident. Any argument about the a priori plausibility of these marketing premises is purposeless. The outcome of the predictive test can, in principle, settle the argument. Rather, the background and rationale for the premises are provided to aid in understanding the present interpretation of the premises and to assist future researchers in formulating and testing alternative premises. The fact that the premises have not been shown false suggests that even sharper restrictions might have been placed on the underlying postulates. From the point of view of improving and sharpening theories of marketing processes, this may be the primary significance of a successful predictive test.

The data includes unit sales, dollar sales, per cent distribution, and advertising expenditures for every brand of the major firms in the industry bimonthly for 16 yr (ninety-six observations). For a shorter period of time, advertising expenditures broken down by media were available. The source of these data was the A. C. Nielsen Co. Promotion data and other supplementary information on the industry were furnished by the Leo Burnett Co.—Advertising.

An analysis of the industry will provide the foundation for the construction of an explanatory marketing model of sales and advertising. This discussion will provide the rationale for decisions about the structural system to be tested and the exogenous and endogenous variables to be included in the structural relations. The following remarks should be read with the understanding that they apply to the historical period studied.

The product analyzed belongs to the general class of frequently purchased products sold predominantly in supermarkets. The product is at the stage of 'innovative maturity'. Innovative maturity has been defined as the stage in which significant new sub-categories are introduced and account for all or most of the sales increases. Unit sales of the product have increased at an average annual rate of growth of 4.5 per cent. Consumption of the product is seasonal.

A handful of firms dominate the industry. Each firm closely follows the actions of competitors and reacts quickly to any significant change in advertising, price, or quality.

The most formidable barrier to entry in the industry is the high absolute cost of product differentiation. New brands require considerable development effort because of technological
complexity and uncertainty in consumer acceptance. Research and development costs expressed as a percentage of sales are far above the average for the food and kindred products industries combined. Although brand loyalty is not especially high, consumers seem to prefer products by established companies. Private labeling is negligible.

Firms introducing a new brand believe that substantial advertising expenditures must be made or the brand will fail. New brands are operationally defined as brands on the market for less than one year. Initially the most important factor is getting the retailer to stock the brand (distribution). Firms follow a 'pull' strategy. Faced with consumer demand for a heavily advertised brand, retailers are forced to stock it. To compensate a retailer for the additional work of listing and displaying new products (new brands or sizes), an introductory allowance of 25-30 cents per case is given for a 3-4 week period. Trade allowances are seldom given at any other time.

Over the period studied, the ratio of advertising to consumer dollar sales for new products has approximately doubled. In a recent interval, this ratio for new brands was about three times that for established brands.

Price competition is avoided by the firms. Industry structure is such that a price cut by one firm would be quickly responded to by its competitors. Changes in market conditions do not cause prices to fluctuate widely. The firms are of the opinion that an across the board price cut probably would not expand the total market enough to increase profits. Demand for the product is believed to be relatively price inelastic. In addition, cross-elasticity of demand between the product and substitute products is negligible, hence firms prefer not to compete on price.

Thus price was believed not to be an important explanatory variable. This belief was supported by the construction of models that included the logarithm of the ratio of the price per unit divided by the food at home index as a variable. The variance of the series for the ratio for a brand and for the remainder were extremely small. In addition, the ratio for a brand was highly correlated with the ratio for the 'remainder'. The remainder is defined as the total market minus the brand minus the sum of all new brands. Perhaps the short-run price effect was hidden by smoothing in the original data collection process.

Sales promotion is relatively unimportant in the industry's marketing mix. It assumes a lesser function because the product is highly differentiated (physically and psychologically). Preceding the historical period studied, the industry went heavily into one particular type of premium. The firms discovered, however, that although such premiums caused some increase in sales, sales declined to their previous levels as soon as the promotion ended. We experimented with a model that included a promotion variable, but have rejected it. It is almost certainly true that there is some influence of promotion, but since promotions tend to be offsetting and sporadic, we consign these effects to the disturbance term in the statistical model.

Consideration of results of price and promotion competition led the firms to emphasize advertising in the marketing mix. Advertising expenditures are high and continuing to increase. While some of the increase comes from increasing advertising time and space expenses, most of it results from greater importance being placed on advertising. An analysis of advertising/sales ratios show that it increased from 0.123 to 0.152 over the period.

Firms appeal directly to consumers through media advertisements. Recently, the firms have been spending approximately 85 per cent of their advertising expenditures on television. They believe advertising to be the most economical and efficient way to (1) make the consumer...
aware of the existence of new brands, (2) force distribution of new brands, and (3) provide the consumer a reason for buying the product.

Advertising by other firms exerts a strong influence on the level of any one firm's expenditures. Firms tend to increase their expenditures to maintain parity with the competition. The level of advertising expenditures is also influenced by the degree of product differentiation and the stage of the product life cycle.

Each firm emphasizes its leading brand by spending a disproportionate amount for its advertising. The firms believe that the image of their leading brand has a favorable influence on the rest of their product line. For this and other reasons the firms attempt to maintain the market share of their leading brands. This and new brand introductions result in a lower than expected expenditure for some other brands in the product line.

AN OVERVIEW

A brief overview of the formal model is provided in order to indicate the general procedure to be used in the predictive test. Employing endogenous and predetermined (lagged endogenous and exogenous) variables, representations, called structural functions, of the individual parts of the market mechanism are developed in isolation from each other. Theoretical marketing premises impose restrictions on the parameters of the structural functions. These restrictions provide the basis for the test of the model. Next, these individual structural functions are assembled into a system of simultaneous equations termed the structural equations.

This system of structural equations can be transformed into a logically equivalent system of reduced-form equations in which each of the endogenous variables is individually related to only the predetermined variables. The limits on parameters of these reduced-form equations can be deduced from the restriction on the parameters of the structural equations. These implied limits on the parameters in conjunction with the estimates of the parameters compose the test of the model.

Based on the description of the industry, a dynamic simultaneous equation model of sales and advertising is constructed. Not only do the advertising and sales of a brand interact with each other, but also they interact with the sales and advertising of all other brands. Thus a model with competition between a brand and the remainder is postulated. The new brands are operationally defined as brands on the market less than one year.

Current sales of a brand result from the contemporaneous influences of the brand's own current advertising and of current advertising of the remainder. Current sales of the brand are also affected by past advertising effort and sales performance of both the brand and the remainder. This past effort is reflected in the sales position of both the brand and the remainder during the previous period. The dynamic feature of the model stems from this introduction of variables lagged one period. Current sales and advertising of 'new brands' also influence current sales of a brand by creating interest in the product and by competing with the brand. Since the product has seasonal characteristics, the level of current sales of the brand depends on the time of year. A similar line of reasoning applies to current sales of the remainder.

The current advertising of a brand depends on past levels of advertising and the effectiveness of this advertising when translated into sales. Sales also generate the stream of dollars necessary to pay for the advertising. The current advertising decision is also highly contingent
on the sales and advertising of the remainder. As with current sales, seasonal influences are present. A similar line of reasoning applies to current advertising of the remainder.

A four equation system is built around the current sales and advertising of the brand and the current sales and advertising of the remainder, incorporating the interdependencies between them. The variables in this system are defined as the logarithm (base = 10) of their per capita value. The coefficient of a variable in the model is called its elasticity and satisfies the usual definition of this term.

The advertising of new brands is treated as an exogenous variable because new brands have introductory strategies and goals that result in 'fixed' high levels of expenditures. Treating sales of new brands as exogeneous avoids the difficult task of explaining the success or failure of a new brand. Distribution is not included as a variable because the level of distribution stabilizes rapidly, especially for successful brands. A rapid increase in distribution occurs during the interval covered by the definition of new brands; consequently, the level of distribution provides negligible additional information relating to the current sales of an established brand. As previously discussed, price and promotion are not considered useful explanatory variables. Note that data on these excluded variables are available; therefore the lack of data has not forced exclusion of these variables.

The quality of advertising is viewed as a sequence of variations occurring about some long-run mean level. This mean level of effectiveness is reflected by the intercept and advertising coefficient terms in a structural equation while the variations are part of the random disturbance terms. The intercept term is also a conglomerate of other factors including the intrinsic quality of the product and a reflection of the segmentation in the market.

![Graph](image)

Fig. 1. Sales and advertising of brand RH2.
A model for one brand (brand RH2) will be formally developed. In this and the following sections brand RH2 will be analyzed in detail. Brand RH2 is a regular brand directed at the ‘high’ category of the most important demographic dimension of the market. It is a leading brand in the industry. In Fig. 1 the time series sales and advertising for brand RH2 are shown.

STRUCTURAL RELATIONS AND POSTULATES

Structural relations are representations of autonomous sectors. Each relation describes the behavior of one sector in isolation from the other sectors. This description consists of postulates made about the form of these structural relations and the constraints imposed upon the parameters of the structural relation. Formally, four autonomous sectors are postulated.

**Proposition 1 (postulate)**

The logarithm of per capita unit sales for a brand can be approximated by the form

$$ S_{B,t} = \beta_{12} A_{B,t} + \beta_{14} A_{R,t} + \gamma_{11} S_{B,t-1} + \gamma_{13} S_{R,t-1} $$

$$ + \gamma_{15} S_{N,t} + \gamma_{16} A_{N,t} + \gamma_{17} D_1 + \gamma_{18} D_2 + \gamma_{19} D_3 $$

$$ + \gamma_{110} D_4 + \gamma_{111} D_5 + \gamma_{112} + u_t $$

(1a)

where the continuous random disturbances $u_t$ are normally distributed with

$$ E(u_t) = 0 \quad \text{and} \quad \text{Var}(u_t) = E(u_t^2) = \delta_u^2 $$

(1b–c)

The symbol $S_{B,t}$ represents the logarithm of per capita unit sales of brand RH2. $A_{B,t}$ represents the logarithm of per capita advertising of brand RH2. $A_{R,t}$ represents the logarithm of per capita advertising of the remainder. The remainder is defined as the Total Market less brand and less New brands. New brands are operationally defined as brands on the market for less than one year. $S_{R,t}$ represents the logarithm of per capita unit sales of the remainder. $S_{N,t}$ represents the logarithm of per capita unit sales of all New brands. Similarly, $A_{N,t}$ represents the logarithm of per capita advertising of all New brands. The dummy variables $D_1, \ldots, D_5$ reflect the seasonal variations. For instance, $D_1 = 1$ in periods 1, 7, 13, \ldots and $= 0$, otherwise. The effect of the dummy variables is to shift the intercept.

**Proposition 2 (postulate)**

Sales of brand RH2 are very inelastic (positive) with respect to its own current advertising

$$ 0.01 < \beta_{12} < 0.04 $$

(2a)

and inelastic (positive) with respect to current advertising of the remainder

$$ 0.00 < \beta_{14} < 0.30 $$

(2b)

and inelastic (positive) with respect to its own lagged sales

$$ 0.60 < \gamma_{11} < 0.80 $$

(2c)

and inelastic (negative) with respect to lagged sales of the remainder

$$ -0.60 < \gamma_{12} < -0.20 $$

(2d)

and finally

$$ 1 \times 10^{-4} < \gamma_u^2 < 4 \times 10^{-4} $$

(2e)
\[ 8 \times 10^{-6} < \gamma_{uv} < 2 \times 10^{-3} \]
\[ 1 \times 10^{-5} < \gamma_{uw} < 5 \times 10^{-4} \]
\[ 8 \times 10^{-5} < \gamma_{ux} < -5 \times 10^{-5}. \]

No hypotheses are made about \( \gamma_{1j}, j = 5, \ldots, 12. \)

The current sales of brand RH2 are postulated to be positively related to current advertising and lagged sales of the brand. Furthermore, current advertising of the remainder, by stimulating primary demand for the product, is assumed to positively aid the individual brand, brand RH2. Reflecting the past advertising effort of the competition and unavailability of a portion of the potential sales because of habit, the lagged sales cross-elasticity is assumed to be negative.

**Proposition 3 (postulate)**

The logarithm of per capita advertising of a brand can be approximated by the form

\[ A_{B,t} = \gamma_{21} S_{B,t-1} + \gamma_{22} A_{B,t-1} + \gamma_{23} S_{R,t-1} + \gamma_{24} A_{R,t-1} + \gamma_{27} D_1 + \gamma_{28} D_2 + \gamma_{29} D_3 + \gamma_{210} D_4 + \gamma_{211} D_5 + \gamma_{212} + \nu_t \]

where the continuous random disturbances \( \nu_t \) are normally distributed with

\[ E(\nu_t) = 0 \quad \text{and} \quad \text{Var}(\nu_t) = E(\nu_t^2) = \delta_{\nu^2}. \]

**Proposition 4 (postulate)**

Advertising of brand RH2 is inelastic (positive) with respect to its own lagged sales

\[ 0.20 < \gamma_{21} < 0.60 \]

and inelastic (positive) with respect to its own lagged advertising

\[ 0.05 < \gamma_{22} < 0.60 \]

and elastic (positive) with respect to lagged sales of the remainder

\[ 1.00 < \gamma_{23} < 4.00 \]

and inelastic with respect to lagged advertising of the remainder

\[ -0.20 < \gamma_{24} < 0.60 \]

and finally

\[ 7 \times 10^{-3} < \delta_{u^2} < 2 \times 10^{-2} \]

\[ -5 \times 10^{-4} < \delta_{uv} < -1 \times 10^{-5} \]

\[ 2 \times 10^{-4} < \delta_{ux} < 2 \times 10^{-3}. \]

No hypotheses are made about \( \gamma_{2j}, j = 7, \ldots, 12. \)

Brand RH2's current advertising is higher than average since it is a leading brand. To reflect this, the brand's own lagged sales elasticity is assumed to be positive. The lagged sales cross-elasticity is postulated to be positive and elastic reflecting the firms' preoccupation with the action of the competition.
Proposition 5 (postulate)

The logarithm of per capita unit sales of the remainder can be approximated by the form

\[ S_{R,t} = \beta_{32} A_{B,t} + \beta_{34} A_{R,t-1} + \gamma_{31} S_{R,t-1} + \gamma_{33} S_{R,t-1} \]

\[ + \gamma_{35} S_{N,t} + \gamma_{36} A_{N,t} + \gamma_{37} D_1 + \gamma_{38} D_2 + \gamma_{39} D_3 + \gamma_{310} D_4 \]

\[ + \gamma_{311} D_5 + \gamma_{312} + w_t \]

where the continuous random disturbances \( w_t \) are normally distributed with

\[ E(w_t) = 0 \quad \text{and} \quad \text{Var}(w_t) = E(w_t^2) = \delta_{w^2}. \]  

Proposition 6 (postulate)

Sales of the remainder are very inelastic with respect to current advertising of brand RH2

\[ -0.02 < \beta_{32} < 0.02 \]  

and inelastic (positive) with respect to its own current advertising

\[ 0.05 < \beta_{34} < 0.15 \]  

and inelastic with respect to lagged sales of the brand RH2

\[ -0.02 < \gamma_{31} < 0.02 \]  

and inelastic (positive) with respect to lagged sales of the remainder

\[ 0.60 < \gamma_{33} < 0.80 \]  

and finally

\[ 4 \times 10^{-5} < \delta_{w^2} < 9 \times 10^{-5} \]  

\[ -9 \times 10^{-5} < \delta_{w^2} < -1 \times 10^{-5}. \]

No hypotheses are made about \( \gamma_{3j}, j = 5, \ldots, 12. \)

As with brand RH2, the current sales of the remainder is postulated to be positively related to its own current advertising and lagged sales. The effect of an individual brand’s current advertising and lagged sales is assumed to small.

Proposition 7 (postulate)

The logarithm of per capita advertising of the remainder can be approximated by the form

\[ A_{R,t} = \gamma_{41} S_{B,t-1} + \gamma_{42} A_{B,t-1} + \gamma_{43} S_{R,t-1} + \gamma_{44} A_{R,t-1} \]

\[ + \gamma_{47} D_1 + \gamma_{48} D_2 + \gamma_{49} D_3 + \gamma_{410} D_4 + \gamma_{411} D_5 + \gamma_{412} + x_t \]

where the continuous random disturbances \( x_t \) are normally distributed with

\[ E(x_t) = 0 \quad \text{and} \quad \text{Var}(x_t) = E(x_t^2) = \delta_{x^2}. \]  

Proposition 8 (postulate)

Advertising of the remainder is inelastic (negative) with respect to lagged sales of brand RH2

\[ -0.40 < \gamma_{41} < -0.05 \]
and inelastic with respect to the lagged advertising of brand RH2
\[ -0.05 < \gamma_{42} < 0.05 \] (8b)

and elastic (positive) with respect to its own lagged sales
\[ 1.00 < \gamma_{43} < 1.65 \] (8c)

and inelastic (positive) with respect to its own lagged advertising
\[ 0.50 < \gamma_{44} < 0.65 \] (8d)

and finally
\[ 1.30 \times 10^{-3} < \delta_{21} < 1.60 \times 10^{-3} \] (8e)

No hypotheses are made about \( \gamma_{45}, j = 7, \ldots, 12 \).

Current advertising of the remainder is assumed to be positively related to its own lagged sales and lagged advertising. No assumption is made about the sign of the effect of the brand's advertising on the remainder's current advertising.

**GENERAL COMMENTS ABOUT HYPOTHESIZED ADVERTISING ELASTICITIES**

We have hypothesized that the demand elasticity with respect to its own advertising is somewhat smaller for brand RH2 than the primary demand elasticity with respect to the advertising of the industry. This hypothesis reflects our belief that the demand for brand RH2 is somewhat less responsive to its own advertising than the typical brand in the industry. In postulating that cross-elasticity of demand with respect to industry advertising is non-negative, we indicate an assumption that the current period net effect of primary demand stimulation is at least not unfavorable for brand RH2, although the size of the restriction indicates considerable uncertainty about the magnitude of this effect. In the postulates concerning the two advertising equations we conjecture that the advertising for brand RH2 as well as the advertising for all other brands is very responsive to increases in previous period industry sales.

**THE SIMULTANEOUS EQUATION MODEL**

Employing postulates which describe the connection between the structural relations previously discussed, a simultaneous equation model of sales and advertising is developed. The connection is assumed to be changeable without affecting the parameters of the structural relations. The connection is logically independent of the marketing premises put forward previously.

Observable time series can be viewed as the combination of two components. The first component consists of the systematic regularities and small random irregularities associated with the functioning of the market mechanism. The other component consists of perturbations caused by phenomena external to the system. This can be expressed mathematically as
\[ y_{t,t} = y_{t,t}^* + y_{t,t}^{**} \] (9)

where \( y_{t,t}^* \) = the equilibrium component of \( y_{t,t}^* \)

and \( y_{t,t}^{**} \) = the deviation from the equilibrium component.
Proposition 10 (postulate)

The entries $y_{1,t}, \ldots, y_{4,t}$ of the dynamic equilibrium path satisfy the equations

\begin{align}
- y_{1,t} + \beta_{12} y_{2,t} + \beta_{14} y_{4,t} + \gamma_{11} y_{1,t-1} + \gamma_{13} y_{3,t-1} + \gamma_{15} y_{5,t} + \\
\gamma_{16} z_{6,t} + \gamma_{17} z_{7,t} + \gamma_{18} z_{8,t} + \gamma_{19} z_{9,t} + \gamma_{10} z_{10,t} + \\
\gamma_{11} z_{11,t} + \gamma_{112} + u_t = 0 \quad (10a)
\end{align}

\begin{align}
- y_{2,t} + \gamma_{21} y_{1,t-1} + \gamma_{22} y_{2,t-1} + \gamma_{23} y_{3,t-1} + \gamma_{24} y_{4,t-1} + \\
\gamma_{27} z_{7,t} + \gamma_{28} z_{8,t} + \gamma_{29} z_{9,t} + \gamma_{210} z_{10,t} + \gamma_{211} z_{11,t} + \\
\gamma_{212} + \epsilon_t = 0 \quad (10b)
\end{align}

\begin{align}
\beta_{32} y_{2,t} - \gamma_{3,t} + \beta_{34} y_{4,t} + \gamma_{31} y_{1,t-1} + \gamma_{33} y_{3,t-1} + \gamma_{35} z_{5,t} + \\
\gamma_{36} z_{6,t} + \gamma_{37} z_{7,t} + \gamma_{38} z_{8,t} + \gamma_{39} z_{9,t} + \gamma_{310} z_{10,t} + \\
\gamma_{311} z_{11,t} + \gamma_{312} + w_t = 0 \quad (10c)
\end{align}

\begin{align}
- y_{4,t} + \gamma_{41} y_{1,t-1} + \gamma_{42} y_{2,t-1} + \gamma_{43} y_{3,t-1} + \gamma_{44} y_{4,t-1} + \\
\gamma_{47} z_{7,t} + \gamma_{48} z_{8,t} + \gamma_{49} z_{9,t} + \gamma_{410} z_{10,t} + \gamma_{411} z_{11,t} + \\
\gamma_{412} + x_t = 0 \quad (10d)
\end{align}

where

- $y_{1,t} =$ log of per capita unit sales of brand RH2
- $y_{2,t} =$ log of per capita advertising of brand RH2
- $y_{3,t} =$ log of per capita unit sales of the remainder
- $y_{4,t} =$ log of per capita advertising of the remainder
- $z_{5,t} =$ log of per capita unit sales of the new brands
- $z_{6,t} =$ log of per capita advertising of the new brands
- $z_{i,t}, i = 7, \ldots, 11,$ = dummy variables for seasonality.

In matrix notation this can be written as

$$
\beta y_t + \gamma \tilde{z}_t + \tilde{u}_t = 0 \quad (11)
$$

or

\[
\begin{bmatrix}
-1 & \beta_{12} & 0 & \beta_{14} \\
0 & -1 & 0 & 0 \\
0 & \beta_{32} & -1 & \beta_{34} \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
y_{1,t} \\
y_{2,t} \\
y_{3,t} \\
y_{4,t}
\end{bmatrix}
+ 
\begin{bmatrix}
\gamma_{11} & 0 & \gamma_{13} & 0 & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} & \gamma_{19} & \gamma_{110} & \gamma_{111} & \gamma_{112} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & 0 & 0 & \gamma_{27} & \gamma_{28} & \gamma_{29} & \gamma_{210} & \gamma_{211} & \gamma_{212} \\
\gamma_{31} & 0 & \gamma_{33} & 0 & \gamma_{35} & \gamma_{36} & \gamma_{37} & \gamma_{38} & \gamma_{39} & \gamma_{310} & \gamma_{311} & \gamma_{312} \\
\gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} & 0 & 0 & \gamma_{47} & \gamma_{48} & \gamma_{49} & \gamma_{410} & \gamma_{411} & \gamma_{412}
\end{bmatrix}
\begin{bmatrix}
y_{1,t-1} \\
y_{2,t-1} \\
y_{3,t-1} \\
y_{4,t-1} \\
z_{5,t} \\
z_{6,t} \\
z_{7,t} \\
z_{8,t} \\
z_{9,t} \\
z_{10,t} \\
z_{11,t} \\
z_{12,t}
\end{bmatrix}
= 0.
\]
REDUCED-FORM EQUATIONS

This system of simultaneous equations can be transformed into a logically equivalent system of equations termed the reduced-form equations. A reduced-form equation contains only one endogenous variable. The reduced-form may be written in matrix notation as

\[ y_t = -\beta^{-1} \gamma z_{t-1} - \beta^{-1} \bar{u}_t. \] (12)

The reduced form can be partitioned and rearranged as

\[ y_t = \Theta y_{t-1} + \pi z_t + u_t. \] (13)

\textit{Proposition 14 (theorem)}

The equilibrium component entries \( y_{1,t}, i = 1, \ldots, 4 \), can be expressed by

\[ y_{1,t} = (\gamma_{11} + \beta_{12} \gamma_{21} + \beta_{14} \gamma_{41}) y_{1,t-1} + (\beta_{12} \gamma_{22} + \beta_{14} \gamma_{42}) y_{2,t-1} + (\gamma_{13} + \beta_{12} \gamma_{23} + \beta_{14} \gamma_{43}) y_{3,t-1} + (\beta_{12} \gamma_{24} + \beta_{14} \gamma_{44}) y_{4,t-1} + \gamma_{15} z_{5,t} + \gamma_{16} z_{6,t} + \gamma_{17} z_{7,t} + \gamma_{18} z_{8,t} + \gamma_{19} z_{9,t} + \gamma_{20} z_{10,t} + \gamma_{21} y_{211} + \beta_{14} \gamma_{411} z_{11,t} + \gamma_{22} y_{212} + \beta_{14} \gamma_{412} + \mu_1,t \] (14a)

\[ y_{2,t} = \gamma_{21} y_{1,t-1} + \gamma_{22} y_{2,t-1} + \gamma_{23} y_{3,t-1} + \gamma_{24} y_{4,t-1} + (\Theta z_{5,t} + (0) z_{6,t} + \gamma_{27} z_{7,t} + \gamma_{28} z_{8,t} + \gamma_{29} z_{9,t} + \gamma_{210} z_{10,t} + \gamma_{211} z_{11,t} + \gamma_{212} + \mu_2,t \] (14b)

\[ y_{3,t} = (\beta_{32} y_{211} + \gamma_{31} + \beta_{34} \gamma_{41}) y_{1,t-1} + (\beta_{32} \gamma_{22} + \beta_{34} \gamma_{42}) y_{2,t-1} + (\beta_{32} \gamma_{23} + \gamma_{33} + \beta_{34} \gamma_{43}) y_{3,t-1} + (\beta_{32} \gamma_{24} + \beta_{34} \gamma_{44}) y_{4,t-1} + \gamma_{35} z_{5,t} + \gamma_{36} z_{6,t} + \gamma_{37} z_{7,t} + \gamma_{38} z_{8,t} + \gamma_{39} z_{9,t} + \gamma_{310} z_{10,t} + \gamma_{311} z_{11,t} + \gamma_{312} + \mu_3,t \] (14c)

\[ y_{4,t} = \gamma_{41} y_{1,t-1} + \gamma_{42} y_{2,t-1} + \gamma_{43} y_{3,t-1} + \gamma_{44} y_{4,t-1} + (0) z_{5,t} + (0) z_{6,t} + \gamma_{47} z_{7,t} + \gamma_{48} z_{8,t} + \gamma_{49} z_{9,t} + \gamma_{410} z_{10,t} + \gamma_{411} z_{11,t} + \gamma_{412} + \mu_4,t \]
where
\[
\begin{align*}
\mu_{1,t} &= \mu_t + \beta_{12} \nu_{t} + \beta_{14} x_{t} \\
\mu_{2,t} &= \nu_{t} \\
\mu_{3,t} &= \beta_{32} \nu_{t} + w_{t} + \beta_{34} x_{t} \\
\mu_{4,t} &= x_{t}
\end{align*}
\]
and the random disturbance vectors \(\mu_t\) are normally distributed with
\[
E(\mu_t) = 0 \tag{14e}
\]
\[
E(\mu_t, \mu_t') = 
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}
\end{bmatrix}
\tag{14f}
\]
where
\[
\begin{align*}
\sigma_{11} &= \delta_{u^2} + 2\beta_{12} \delta_{u\nu} + 2\beta_{14} \delta_{u\nu} + 2\beta_{12} \beta_{14} \delta_{\nu\nu} + \beta_{12} \delta_{x^2} + \\
&+ \beta_{14} \delta_{x^2} \\
\sigma_{12} &= \delta_{u\nu} + \beta_{12} \delta_{\nu\nu} + \beta_{14} \delta_{\nu x} \\
\sigma_{13} &= \beta_{32} \delta_{u\nu} + \beta_{12} \beta_{32} \delta_{x^2} + (\beta_{12} \beta_{34} + \beta_{14} \beta_{32}) \delta_{\nu x} + \delta_{u x} \\
&+ \beta_{12} \delta_{\nu x} + \beta_{14} \delta_{x x} + \beta_{34} \delta_{x x} + \beta_{14} \beta_{34} \delta_{x^2} \\
\sigma_{14} &= \delta_{u x} + \beta_{12} \delta_{x x} + \beta_{34} \delta_{x^2} \\
\sigma_{22} &= \delta_{v^2} \\
\sigma_{23} &= \beta_{32} \delta_{v\nu} + \delta_{v\nu} + \beta_{34} \delta_{v x} \\
\sigma_{24} &= \delta_{v x} \\
\sigma_{33} &= \beta_{32} \delta_{v^2} + 2\beta_{32} \delta_{v\nu} + 2\beta_{32} \beta_{34} \delta_{v x} + 2\beta_{34} \delta_{v x} + \delta_{v^2} + \beta_{34} \delta_{x^2} \\
\sigma_{34} &= \beta_{32} \delta_{v x} + \delta_{v x} + \beta_{34} \delta_{x^2} \\
\sigma_{44} &= \delta_{x^2}.
\end{align*}
\tag{14g-14p}
\]

THE FINAL FORM AND STABILITY CONDITIONS

The reduced form can be transformed further so that each endogenous variable is described solely in terms of exogenous variables and its own lagged values. Thus a final-form equation contains one endogenous variable and does not contain current or lagged values of any other endogenous variables.

The final form can be expressed in matrix notation as:
\[
y_t = \sum_{\tau=0}^{\infty} \theta^\tau \pi_{z,t-\tau} + \sum_{\tau=0}^{\infty} \nu_{t-\tau} \tag{15}
\]
assuming that \(\lim_{\tau \to 0} \theta^\tau = 0\), see Goldberger (1964).
If the condition that \( \lim_{t \to 0} \theta^r = 0 \) is satisfied, the system is said to be stable. This stability condition requires that each characteristic root of \( \theta \) be less than one in absolute value.

Thus the stability conditions impose these additional restrictions on the structural model:

\[
C (\lambda = 1) > 0 \quad \text{and} \quad C (\lambda = -1) > 0
\]  

(16 – 17)

where \( \lambda \) is a characteristic root of \( \theta \).

From the restrictions on the \( \beta \)'s and \( \gamma \)'s postulated previously, deduction shows that the stability conditions are not intrinsically satisfied\(^1\). Thus the stability conditions are, in fact, an additional restriction on the model

\[
-0.219 < C (\lambda = 1) < 0.269 \quad \text{(18a)}
\]

and

\[
2.94 < C (\lambda = -1) < 10.93. \quad \text{(18b)}
\]

THE MAINTAINED HYPOTHESIS

The maintained hypothesis is the class of alternative hypotheses against which the postulates previously introduced are to be tested.

Proposition 19 (maintained hypothesis)

(a) The equilibrium paths \( y_{1,t} \), \( y_{2,t} \), \( y_{3,t} \), \( y_{4,t} \) are defined by the system of difference equations.

\[
y_{1,t} = x_{11} y_{1,t-1} + x_{12} y_{2,t-1} + x_{13} y_{3,t-1} + x_{14} y_{4,t-1} + x_{15} z_{5,t-1} + x_{16} z_{6,t-1} + x_{17} z_{7,t-1} + x_{18} z_{8,t-1} + x_{19} z_{9,t} + x_{20} z_{10,t-1} + x_{21} z_{11,t-1} + x_{22} z_{12,t-1} + e_{1,t} \quad \text{(19a)}
\]

\[
y_{2,t} = x_{21} y_{1,t-1} + x_{22} y_{2,t-1} + x_{23} y_{3,t-1} + x_{24} y_{4,t-1} + x_{25} z_{5,t-1} + x_{26} z_{6,t-1} + x_{27} z_{7,t-1} + x_{28} z_{8,t-1} + x_{29} z_{9,t} + x_{30} z_{10,t-1} + x_{31} z_{11,t-1} + x_{32} z_{12,t-1} + e_{2,t} \quad \text{(19b)}
\]

\[
y_{3,t} = x_{31} y_{1,t-1} + x_{32} y_{2,t-1} + x_{33} y_{3,t-1} + x_{34} y_{4,t-1} + x_{35} z_{5,t-1} + x_{36} z_{6,t-1} + x_{37} z_{7,t-1} + x_{38} z_{8,t-1} + x_{39} z_{9,t} + x_{40} z_{10,t-1} + x_{41} z_{11,t-1} + x_{42} z_{12,t-1} + e_{3,t} \quad \text{(19c)}
\]

\[
y_{4,t} = x_{41} y_{1,t-1} + x_{42} y_{2,t-1} + x_{43} y_{3,t-1} + x_{44} y_{4,t-1} + x_{45} z_{5,t-1} + x_{46} z_{6,t-1} + x_{47} z_{7,t-1} + x_{48} z_{8,t-1} + x_{49} z_{9,t} + x_{50} z_{10,t-1} + x_{51} z_{11,t-1} + x_{52} z_{12,t-1} + e_{4,t} \quad \text{(19d)}
\]

where the random disturbance vectors \( e_{1,t} \) are normally distributed with

\[
E (e_{1,t}) = 0
\]

\[
E (e_{1,t}, e_{1,t}') = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}
\end{bmatrix}
\]  

\[(19e)\]

\[1. \text{ Calculated using the Research Analysis Corporation's Computer Program Implementing the Sequential Unconstrained Minimization Technique for Non-linear Programming.}\]
(b) The entries $y_{i,t}, i = 1, \ldots, 4$, can be represented by final-form equations.

As with the simultaneous equations model, this regression model is stable if the matrix of coefficients of the lagged variables has only characteristic roots less than one.

THE IDENTIFIABILITY OF THE STRUCTURAL MODEL

The premises underlying the simultaneous equations model, proposition 10, have been postulated to predict the behavior of statistical estimates of the parameters of the maintained regression model, proposition 19. The regression equations (19a–d) and the reduced-form equations (14a–d) are hypothesized to be identical in all of their parts. The identifiability condition is a statement that the parameters of the regression model can be expressed as explicit functions of the parameters of the structural model. The fulfillment of the identifiability condition is testable, see Basmann (1965b).

Proposition 20 (postulate)

The identifiability functions $\phi_i$ vanish at the true parameter point.

This can be expressed in matrix notation as

$$
\begin{align*}
\beta A + \gamma &= 0 \\
\beta \Sigma \beta' - \Delta &= 0.
\end{align*}
$$

(20a)

Written out, these equations consist of fifty-eight identifiability relations. The model is identifiable if the following restrictions on the regression parameters are satisfied. These restrictions will be tested in a following section.

$$
\begin{align*}
x_{25} &= 0, & x_{26} &= 0, & x_{45} &= 0, & x_{46} &= 0.
\end{align*}
$$

(21a–d)

PARAMETER SPACE AND PREDICTIONS FOR STATISTICAL TEST

The representation described by proposition 10 is termed the structural model. The symbol $B$ denotes the structural parameter space. The symbol $\beta$ denotes a typical point belonging to $B$. Thus $B$ is defined by all points $\beta$ that satisfy the inequalities (2a–h), (4a–g), (6a–f), (8a–e), and (18a–b).

The representation described by proposition 19 is termed the regression model. The symbol $A$ denotes the regression parameter space. The symbol $\alpha$ denotes a typical point belonging to $A$. Satisfaction of the identifiability conditions means that the structural parameter space $B$ may be mapped onto a subset of the regression parameter space $A$. Instead of constructing this subset of $A$, which is a complicated figure, a rectangle set based on the implied minimum and maximum values of the regression parameters is constructed. This rectangle set is termed the weaker hypothesis. If the weaker hypothesis is in poor agreement with the observations, then the strict hypothesis is also in poor agreement with the observations. The permissible signs and magnitudes of the regression parameters have been derived and are shown later in Table 1. These inequalities are the foundation for the predictive test. Additional tests are imposed by the dynamic stability and identifiability conditions.

THE STATISTICAL RESULTS

Initial conditions

The predictive regression test was carried out on the assumption that the external initial conditions for the measures of the market ($y^*_t, t = 1, \ldots, 4$) were approximately constant.
over the historical period analyzed. In addition, the initial deviations from the equilibrium paths \((y^*_{t,0}, i = 1, \ldots, 4)\) were assumed to be negligible relative to the corresponding standard deviations of \(e_{t,i}, i = 1, \ldots, 4\). This means that the market observations \((y^*_{t,i})\) and their corresponding dynamic equilibrium components \((y_{t,i})\) are equivalent.

**Predictive test**

Statistical estimates\(^2\) of the regression parameters (proposition 19) are exhibited below:

\[
y_{1,t} = 0.736 \, y_{1,t-1} + 0.010 \, y_{2,t-1} - 0.069 \, y_{3,t-1} - 0.004 \, y_{4,t-1} \tag{22a}
\]

(i) 
\[
(0.083) \quad (0.016) \quad (0.099) \quad (0.035)
\]

(ii) 
\[
(8.81) \quad (0.594) \quad (-0.696) \quad (-0.120)
\]

\[
+ 0.005 \, z_{5,t} - 0.006 \, z_{6,t} + 0.035 \, z_{7,t} + 0.029 \, z_{8,t} + 0.004 \, z_{9,t}
\]

(i) 
\[
(0.004) \quad (0.003) \quad (0.005) \quad (0.008) \quad (0.010)
\]

(ii) 
\[
(1.29) \quad (-1.79) \quad (6.73) \quad (3.69) \quad (0.351)
\]

\[
- 0.043 \, z_{10,t} - 0.036 \, z_{11,t} - 0.247 \quad \text{(iii)} \quad R = 0.945
\]

(i) 
\[
(0.010) \quad (0.006) \quad (0.096)
\]

(ii) 
\[
(-4.41) \quad (-6.16) \quad (-2.56)
\]

\[
y_{2,t} = 0.569 \, y_{1,t-1} + 0.274 \, y_{2,t-1} + 2.57 \, y_{3,t-1} - 0.141 \, y_{4,t-1} \tag{22b}
\]

(i) 
\[
(0.594) \quad (0.117) \quad (0.708) \quad (0.250)
\]

(ii) 
\[
(0.985) \quad (2.34) \quad (3.63) \quad (-0.565)
\]

\[
+ 0.032 \, z_{5,t} - 0.028 \, z_{6,t} - 0.043 \, z_{7,t} - 0.203 \, z_{8,t} - 0.306 \, z_{9,t}
\]

(i) 
\[
(0.029) \quad (0.025) \quad (0.037) \quad (0.057) \quad (0.072)
\]

(ii) 
\[
(1.11) \quad (-1.13) \quad (-1.16) \quad (-3.58) \quad (-4.23)
\]

\[
- 0.309 \, z_{10,t} - 0.152 \, z_{11,t} - 0.843 \quad \text{(iii)} \quad R = 0.850
\]

(i) 
\[
(0.070) \quad (0.152) \quad (0.686)
\]

(ii) 
\[
(-4.40) \quad (-3.67) \quad (-1.23)
\]

\[
y_{3,t} = -0.015 \, y_{1,t-1} + 0.004 \, y_{2,t-1} + 0.847 \, y_{3,t-1} + 0.044 \, y_{4,t-1} \tag{22c}
\]

(i) 
\[
(0.045) \quad (0.009) \quad (0.053) \quad (0.019)
\]

(ii) 
\[
(-0.323) \quad (0.465) \quad (15.9) \quad (23.5)
\]

\[
+ 0.003 \, z_{5,t} - 0.003 \, z_{6,t} + 0.021 \, z_{7,t} - 0.001 \, z_{8,t} - 0.017 \, z_{9,t}
\]

(i) 
\[
(0.002) \quad (0.002) \quad (0.003) \quad (0.004) \quad (0.005)
\]

(ii) 
\[
(1.17) \quad (-1.50) \quad (7.51) \quad (-0.346) \quad (-3.07)
\]

\[
- 0.049 \, z_{10,t} - 0.036 \, z_{11,t} + 0.048 \quad \text{(iii)} \quad R = 0.993
\]

(i) 
\[
(0.005) \quad (0.003) \quad (0.052)
\]

(ii) 
\[
(-9.34) \quad (-11.4) \quad (0.930)
\]

\[
y_{4,t} = -0.378 \, y_{1,t-1} + 0.036 \, y_{2,t-1} + 1.20 \, y_{3,t-1} + 0.514 \, y_{4,t-1} \tag{22d}
\]

(i) 
\[
(0.240) \quad (0.047) \quad (0.286) \quad (0.101)
\]

\(-0.378 \, y_{1,t-1} + 0.036 \, y_{2,t-1} + 1.20 \, y_{3,t-1} + 0.514 \, y_{4,t-1}\)
(ii) 
\[ (-1.57) (0.768) (4.20) (5.08) \]
\[ -0.002 z_{5,t} + 0.001 z_{6,t} + 0.001 z_{7,t} - 0.077 z_{8,t} - 0.117 z_{9,t} \]

(i) 
\[ (0.012) (0.010) (0.015) (0.023) (0.029) \]

(ii) 
\[ (-0.176) (0.123) (0.720) (-3.36) (-4.00) \]
\[ -0.069 z_{10,t} - 0.103 z_{11,t} - 0.728 \]

(iii) \[ R = 0.971 \]

(i) 
\[ (0.028) (0.017) (0.277) \]

(ii) 
\[ (-2.44) (-6.16) (-2.63) \]

where

(i) exhibits estimates of the large sample standard deviation for each coefficient estimate
(ii) exhibits the ratio of coefficient estimate to estimated standard deviation
(iii) exhibits the sample multiple regression coefficient

and

\[ \hat{\Sigma} = \begin{bmatrix}
1.77 \times 10^{-4} & 3.28 \times 10^{-4} & 2.23 \times 10^{-3} & -1.99 \times 10^{-5} \\
8.97 \times 10^{-3} & 7.99 \times 10^{-5} & 9.72 \times 10^{-4} \\
5.12 \times 10^{-5} & 8.39 \times 10^{-5} & 1.47 \times 10^{-4}
\end{bmatrix} \]  \hspace{1cm} (22e)

The exact finite sample distributions of the estimates shown in (22a–e) are unknown. Thus the customary use of ‘t-ratios’ as statistics for testing the significance of the coefficients may not be appropriate. However, the conjecture is made that the maximum likelihood estimators of the regression coefficients are approximately jointly normally distributed under the initial conditions previously described.

**Identifiability test**

Utilizing the conjecture, the restrictions (21a–d) are tested using a critical region of size \( \alpha = 0.05 \). To test the first restriction \( (z_{25} = 0) \), the following test statistic was used

\[ t'_{z_{25}} = \frac{z_{25}}{\sqrt{\hat{\Sigma}^{55} \sigma_{z_{25}}}} \]  \hspace{1cm} (23)

where \( \hat{\Sigma}^{55} \) is the fifth diagonal element of \( (z' \hat{z})^{-1} \). Employing a two tailed test, the restriction \( (z_{25} = 0) \) seems to be in good agreement with the observations. Tests on the other restrictions \( (z_{26} = 0, z_{45} = 0, z_{46} = 0) \) also were non-significant. Thus the structural advertising equations were judged to be identified.

**Dynamic stability**

In the previous discussion it was suggested that the condition for dynamic stability was that each of the roots of the matrix of coefficients of the lagged endogenous variables in the regression model be less than one in absolute value. These roots were each found to be less than one in magnitude. Therefore the regression model is dynamically stable.

**Analysis of predictions**

In addition to the identifiability and stability tests, predictions have been made about the minimum and maximum values of the coefficients of the regression model admissible under the structural premises. These regression estimates are in good agreement with structural premises insofar as the weak hypothesis is concerned.
Residuals and serial correlations of residuals for regression equations

The residuals for the regression equations (19a–d) have been carefully examined. The von Neumann ratios for the four equations are: 2.037, 1.928, 2.028, 2.149. Applying the von Neumann ratio test for autocorrelation (based on a normal approximation), we find that at the 5 per cent level none of these is significant.

Inspection of residuals reveals eight instances where the estimated random disturbance for a given regression equation exceeds twice its estimated standard deviation. This suggests the possibility of the existence of non-negligible deviations from the dynamical equilibrium paths during the historical period analyzed. However, in none of these instances is the disturbance more than three times its estimated standard deviation.

Table 1. Maxima and minima regression parameters admissible under the structural premises and estimates of the regression parameters

<table>
<thead>
<tr>
<th>Minimum Values</th>
<th>Coefficient Estimates</th>
<th>Maximum Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.482</td>
<td>( a_{11} ) (0.736)</td>
<td>0.824</td>
</tr>
<tr>
<td>0.015</td>
<td>( a_{12} ) (0.010)</td>
<td>0.039</td>
</tr>
<tr>
<td>0.590</td>
<td>( a_{13} ) (–0.069)</td>
<td>0.455</td>
</tr>
<tr>
<td>0.008</td>
<td>( a_{14} ) (–0.004)</td>
<td>0.219</td>
</tr>
<tr>
<td>0.150</td>
<td>( a_{21} ) (0.569)</td>
<td>0.600</td>
</tr>
<tr>
<td>0.050</td>
<td>( a_{22} ) (0.274)</td>
<td>0.600</td>
</tr>
<tr>
<td>1.00</td>
<td>( a_{23} ) (2.57)</td>
<td>4.00</td>
</tr>
<tr>
<td>0.200</td>
<td>( a_{24} ) (–0.140)</td>
<td>0.600</td>
</tr>
<tr>
<td>0.092</td>
<td>( a_{31} ) (–0.015)</td>
<td>0.030</td>
</tr>
<tr>
<td>0.020</td>
<td>( a_{32} ) (0.004)</td>
<td>0.020</td>
</tr>
<tr>
<td>0.570</td>
<td>( a_{33} ) (0.847)</td>
<td>1.13</td>
</tr>
<tr>
<td>0.013</td>
<td>( a_{34} ) (0.044)</td>
<td>0.110</td>
</tr>
<tr>
<td>0.400</td>
<td>( a_{41} ) (–0.378)</td>
<td>–0.050</td>
</tr>
<tr>
<td>0.050</td>
<td>( a_{42} ) (0.036)</td>
<td>0.050</td>
</tr>
<tr>
<td>1.00</td>
<td>( a_{43} ) (1.20)</td>
<td>1.65</td>
</tr>
<tr>
<td>0.500</td>
<td>( a_{44} ) (0.510)</td>
<td>0.650</td>
</tr>
<tr>
<td>5.20 \times 10^{-8}</td>
<td>( a_{11} ) (1.77 \times 10^{-4})</td>
<td>6.40 \times 10^{-4}</td>
</tr>
<tr>
<td>7.80 \times 10^{-8}</td>
<td>( a_{12} ) (3.28 \times 10^{-3})</td>
<td>3.40 \times 10^{-3}</td>
</tr>
<tr>
<td>6.80 \times 10^{-8}</td>
<td>( a_{13} ) (2.23 \times 10^{-5})</td>
<td>5.89 \times 10^{-4}</td>
</tr>
<tr>
<td>7.80 \times 10^{-8}</td>
<td>( a_{14} ) (–1.99 \times 10^{-4})</td>
<td>5.10 \times 10^{-4}</td>
</tr>
<tr>
<td>7.00 \times 10^{-8}</td>
<td>( a_{22} ) (8.97 \times 10^{-3})</td>
<td>2.00 \times 10^{-3}</td>
</tr>
<tr>
<td>9.10 \times 10^{-8}</td>
<td>( a_{23} ) (7.99 \times 10^{-6})</td>
<td>5.40 \times 10^{-4}</td>
</tr>
<tr>
<td>2.00 \times 10^{-4}</td>
<td>( a_{24} ) (9.72 \times 10^{-4})</td>
<td>2.00 \times 10^{-3}</td>
</tr>
<tr>
<td>8.55 \times 10^{-8}</td>
<td>( a_{25} ) (5.11 \times 10^{-5})</td>
<td>1.65 \times 10^{-4}</td>
</tr>
<tr>
<td>6.50 \times 10^{-8}</td>
<td>( a_{26} ) (8.39 \times 10^{-5})</td>
<td>1.74 \times 10^{-4}</td>
</tr>
<tr>
<td>1.30 \times 10^{-3}</td>
<td>( a_{43} ) (1.47 \times 10^{-3})</td>
<td>1.60 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Structural estimates

The estimates of the parameters of the structural sales equations in the model for brand RH2 are

\[ y_{1,t} = 0.035 y_{2,t} + 0.001 y_{4,t} + 0.716 y_{1,t-1} - 0.162 y_{3,t-1} \]

\[ + 0.004 z_{5,t} - 0.005 z_{6,t} + 0.036 z_{7,t} + 0.036 z_{8,t} + 0.014 z_{9,t} \]

(24a)
\[ y_{3,t} = 0.003 y_{2,t} + 0.087 y_{4,t} + 0.016 y_{1,t-1} + 0.733 y_{3,t-1} \]  
\[ + 0.002 z_{5,t} - 0.002 z_{6,t} + 0.020 z_{7,t} + 0.006 z_{8,t} - 0.005 z_{9,t} \]
\[ - 0.042 z_{10,t} - 0.026 z_{11,t} + 0.114 \]

where (i) exhibits estimates of the large sample standard deviation for each coefficient estimate

\[
\hat{\Delta} = \begin{bmatrix}
1.65 \times 10^{-4} & 8.85 \times 10^{-6} & 2.43 \times 10^{-5} & -5.58 \times 10^{-5} \\
8.90 \times 10^{-3} & -3.54 \times 10^{-5} & 9.40 \times 10^{-4} \\
4.78 \times 10^{-5} & -4.66 \times 10^{-5} & 1.43 \times 10^{-3}
\end{bmatrix}
\]

Examining the structural estimates, it would appear that the advertising elasticity for this leading brand is small and that it is quite possibly smaller than the advertising elasticity of primary demand for the industry.

FORECASTING

The forecasting ability of the model was examined by using the first eighty-six observations to estimate the parameters of the regression model. These estimates were employed to generate a ten period forecast. Note that in practice management might specify the sequence of its own advertising expenditures in making a forecast.

The forecast might form the basis for an additional test of non-negligible deviations from the equilibrium path. However, in order to conduct such a test a criterion is required which is

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![Figure 2: Ten period forecast—per capita unit sales of brand RH2.](image-url)
Fig. 3. Ten period forecast—per capita unit sales of remainder (RH2).

Fig. 4. Ten period forecast—per capita advertising of brand RH2.

independent of the least squares estimates upon which the forecast is based. An adequate theory of external shocks does not yet exist. Nevertheless, in the hope that future developments will establish the basis for a test, the data are presented here.

Figures 2–5 illustrate the actual and predicted values for each of the endogenous variables in a ten period forecast. The data have been scaled so that the actual value in period 87 equals 100. The model predicts the sequence of observations and the turning points in each of the
Fig. 5. Ten period forecast—per capita advertising of the remainder (RH2).

sales series remarkably well. While the accuracy of a forecast is not a proper basis for accepting or rejecting an explanatory model, the results in this case do lend credibility to the model.

SUMMARY AND CONCLUSIONS

We have constructed a model of sales and advertising which has withstood a rigorous predictive test and an exhaustive examination of underlying statistical premises. While a successful predictive test of a model does not establish conclusively that the model is correct, it does lend substance to the argument for it. Moreover, the fact that the theory has not been falsified provides an incentive for further testing with even sharper restrictions on the basic postulates.

Predictive testing within the framework of simultaneous equation regression models is especially useful in those instances in which structural estimation is difficult, either because of identification or statistical precision. The structural equations parameter estimates for advertising have relatively large standard deviations in the data we have just examined. The predictive test, however, lends credibility to our estimates.

Several interesting conclusions about advertising and sales emerge from our analysis. These conclusions apply, of course, only to the particular brand studied. The advertising elasticity for the brand is small, requiring relatively larger expenditures to offset sales gains by other brands. The advertising expenditures for the brand are extremely responsive to sales increases of other brands. Advertising expenditures do appear, in this case, to stimulate the primary demand. The carry-over or cumulative effects of advertising are significant inasmuch as the lagged influence of sales is substantial.

The forecasting qualities of the model are good. The ability to forecast the influence of advertising on sales as well as the influence of sales of the brand and other brands on advertis-
ing in a system of interdependencies provides the basis for an evaluation of alternative strategies. Continuous predictive testing and model revision suggests the possibility of developing a dynamic management information system that not only furnishes a rigorous explanation of events, but also establishes the basis for normative decision rules that have significant profit potential.

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